Thermodynamics of SU(2) bosons in one dimension

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Abstract. On the basis of Bethe ansatz solution of two-component bosons with SU(2) symmetry and δ-function interaction in one dimension, we study the thermodynamics of the system at finite temperature by using the strategy of thermodynamic Bethe ansatz (TBA). It is shown that the ground state is an isospin "ferromagnetic" state by the method of TBA, and at high temperature the magnetic property is dominated by Curie's law. We obtain the exact result of specific heat and entropy in strong coupling limit which scales like T at low temperature. While in weak coupling limit, it is found there is still no Bose-Einstein Condensation (BEC) in such 1D system.

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A two-component Bose gas has been produced in magnetically trapped ^{87}Rb by rotating the two hyperfine states into each other with the help of slightly detuned Rabi oscillation field [1]. It was noticed [2] that the properties of such Bose system can be different from the traditional scalar Bose system once it acquires internal degree of freedom. Bethe ansatz solution of SU(2) two-component bosons in one dimension was obtained[3, 4]. It was pointed out that the ground state of such a system is an isospin "ferromagnetic" state[4] which differs from that of spin-1/2 fermions in one dimension, such as SU(2) Hubbard model[5] etc..

An interacting SU(2) boson field trapped in a one dimensional ring of length L can be modeled by the following Hamiltonian

$$H = \int dx \left[\sum_a \partial_x \psi_a^* \partial_x \psi_a + \frac{c}{2} \sum_{a,b} \psi_a^* \psi_a \psi_b^* \psi_b \right] \tag{1}$$

where $a, b = 1, 2$ denotes the z-component of isospin. The Bethe ansatz equations (BAE) of eq. (1) are obtained as follows

$$\begin{aligned} e^{ik_jL} &= -\prod_{l=1}^{N} \frac{k_j - k_l + ic}{k_j - k_l - ic} \prod_{\nu=1}^{M} \frac{k_j - \lambda_\nu - ic/2}{k_j - \lambda_\nu + ic/2} \\ 1 &= -\prod_{l=1}^{N} \frac{\lambda_\gamma - k_l - ic/2}{\lambda_\gamma - k_l + ic/2} \prod_{\nu=1}^{M} \frac{\lambda_\gamma - \lambda_\nu + ic}{\lambda_\gamma - \lambda_\nu - ic} \end{aligned} \tag{2}$$

where M denotes the total number of down isospins. Eq. (2) differs from the BAE of scalar bosons with periodic condition[6]. The second equation of eqs. (2) of isospin rapidity λ arises from the application of quantum inverse method[7], which can be inferred from spin-1/2 fermions[8] too. However, the symmetry of bosons wave function leaves the first term on the right side of first equation of eqs. (2), which does not appear in the BAE of fermions.

The strategy we use here is the thermodynamic Bethe ansatz (TBA) which was pioneered by C. N. Yang and C. P. Yang for the case of the delta-function Bose gas[10]. It is used to derive a set of nonlinear integral equations called TBA equations, which describe the thermodynamics of the model at finite temperature. Moreover, the λs can be complex roots which should form a "bound state" with other λs[11] when $T \neq 0$, which arises from the consistency of both sides of the BAE. For ideal λ strings of length m the rapidities are $\Lambda_a^{nj} = \lambda_a^n + (n + 1 - 2j)iu + O(\exp(-\delta N))$. Here $u = c/2$, a enumerates the strings of the same length m, and $j = 1, \ldots, m$ counts the λs involved in the ath λ string of the length m, λ_a^n is the real part of the string.

Taking logarithm of the BAE (2) by using string hypothesis we arrive at the following discrete Bethe ansatz equations

$$\begin{aligned} 2\pi I_j &= k_jL + 2\sum_l \Theta_2(k_j - k_l) - \sum_{an} \Theta_n(k_j - \lambda_a^n) \\ 2\pi J_a^n &= 2\sum_l \Theta_n(\lambda_a^n - k_l) - 2\sum_{bl, t\neq 0} A_{nlt} \Theta_t(\lambda_a^n - \lambda_b^l) \end{aligned} \tag{3}$$

where $\Theta_n(x) = \tan^{-1}(x/nu)$ and

$$A_{nlt} = \begin{cases} 1, & \text{for } t = n+l, |n-l| \\ 2, & \text{for } t = n+l-2, \cdots, |n-l|+2 \\ 0, & \text{otherwise.} \end{cases}$$

I_j and J_a^n play the role of quantum numbers for charge rapidity and isospin rapidity respectively. In order to guarantee linearly independent of wave function, all quantum number within a given set of $\{I\}$ as well as that in $\{J\}$ should be different. An arbitrary quantum number may be in a set (charge roots) or missing (charge hole). In thermodynamic limit, the distribution of charge rapidities becomes dense and it is useful to introduce the density function for charge roots and holes respectively. We denote with $\rho(k)$ and $\rho^h(k)$ the density function of charge roots and holes, in a similar way, with $\sigma_n(\lambda)$ and σ_n^h the density function of n-strings roots and holes on real axis. They are defined by

$$\begin{aligned} \rho(k) + \rho^h(k) = \quad & (1/L)dI(k)/dk \\ \sigma_n(\lambda) + \sigma_n^h(\lambda) = \ & (1/L)dJ^n(\lambda)/d\lambda. \end{aligned} \tag{4}$$

Then from eqs. (3) we obtain a set of coupled integral equations.

$$\begin{aligned} \rho + \rho^h = \quad & \frac{1}{2\pi} + K_2(k) * \rho(k) - \sum_n K_n(k) * \sigma_n(k) \\ \sigma_n + \sigma_n^h = \ & K_n(\lambda) * \rho(\lambda) - \sum_{l,t\neq 0} A_{nlt} K_t(\lambda) * \sigma_l(\lambda) \end{aligned} \tag{5}$$

where $K_n(x) = nu/\pi(n^2u^2 + x^2)$, and $*$ denotes a convolution.

In terms of the density functions of charge and isospin roots, the kinetic energy per length has the form $E_k/L = \int k^2 \rho(k)dk$, the total number of down isospins is $M/L = \sum_n n \int \sigma_n(\lambda)d\lambda$ and the particle density of the model is $D = N/L = \int \rho(k)dk$. If we consider the energy arising from the external field Ω which is the Rabi field in two-component BEC experiments, the internal energy of the model is

$$E/L = \int (k^2 - \Omega)\rho(k)dk + \sum_n 2n\Omega \int \sigma_n d\lambda. \tag{6}$$

And with the help of the approach first introduced by Yang and Yang[10], the entropy of the present model at finite temperature is

$$\begin{aligned} \mathcal{S}/L = & \int [(\rho + \rho^h)\ln(\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h]dk \\ & + \sum_n \int [(\sigma_n + \sigma_n^h)\ln(\sigma_n + \sigma_n^h) - \sigma_n \ln \sigma_n - \sigma_n^h \ln \sigma_n^h]d\lambda. \end{aligned} \tag{7}$$

The Gibbs free energy of the model then is defined by $F = E - T\mathcal{S} - \mu N$, where μ is the chemical potential. In order to obtain the thermal equilibrium, we minimize the free energy with respect to all the density functions subjects to the constraint (5). In addition, the total number of particles, the magnetization are to be keep constant. For

this purpose, the chemical potential μ and external field Ω play the role of Lagrange multipliers.

It is useful to define the energy potential for charge sector and isospin sector:

$$\begin{aligned}\kappa(k) &= e^{\epsilon(k)/T} = \rho^h(k)/\rho(k)\\ \eta_n(\lambda) &= \sigma_n^h(\lambda)/\sigma_n(\lambda).\end{aligned} \tag{8}$$

Applying the minimum condition $\delta F = 0$ gives rise to a revised version of Gaudin-Takahashi equations

$$\begin{aligned}T\ln\kappa = \epsilon(k) &= k^2 - \mu - \Omega - TK_2(k) * \ln[1+\kappa^{-1}]\\ &\quad - T\sum_n K_n(k) * \ln[1+\eta^{-1}]\\ \ln\eta_1 &= \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u) * \ln[(1+\kappa^{-1})(1+\eta_2)]\\ \ln\eta_n &= \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u) * \ln[(1+\eta_{n-1})(1+\eta_{n+1})].\end{aligned} \tag{9}$$

And these equations are completed by the asymptotic conditions

$$\lim_{n\to\infty}[\ln\eta_n/n] = 2x \tag{10}$$

where $x = \Omega/T$. Eqs. (9) can be solved by iteration. Note that eqs. (5) together with eqs. (9) completely determine the densities of charge roots and isospin roots in the state of thermal equilibrium. The Helmholtz free energy $F = E - T\mathcal{S}$ is given by

$$F = \mu N - \frac{TL}{\pi}\int \ln[1+e^{-\epsilon/T}]dk. \tag{11}$$

The above approach called TBA is universal for discussing the thermodynamics of one dimensional integrable model. Once eqs. (9) are solved, all thermodynamic quantities can be obtained from eq. (11) in principle.

Magnetic property: zero and high temperature limit: The state at zero temperature is the ground state. The Fermi surface is determined by $\epsilon(k_F) = 0$. Since there is no hole under Fermi surface, we can take the energy potential $\kappa = \rho^h/\rho$ as zero. As a result, from eqs. (9), it is easy to see $\eta_n \to \infty$, and $M = 0$, the "ferromagnetic" ground state. The first equation of eqs. (9) becomes

$$\epsilon_0(k) = k^2 - \mu - \Omega + K_2(k) * \epsilon_0(k) \tag{12}$$

which gives the solution of dressed energy, and the ground-state energy may be given in terms of ϵ_0

$$E_0/L = \frac{1}{2\pi}\int_{-k_F}^{k_F}\epsilon_0(k)dk. \tag{13}$$

Consequently, the ground state of 1D SU(2) bosons is an isospin "ferromagnetic" state, which coincides with the analysis of Li et al.[4]. Then the property of the model at $T = 0$ is the same as that of scalar bosons in one dimension which has been discussed extensively by Lieb and Liniger[6]. In the isospin space, however, the SU(2) symmetry of whole system around the ground state is broken.

In the high temperature limit $T \to \infty$ (free isospins), however we can assume that all functions $\eta_n(\lambda)$ are independent of λ. Then eqs. (9) can be written as follows,

$$\eta_1^2 = (1+\eta_2)$$
$$\eta_n^2 = (1+\eta_{n-1})(1+\eta_{n+1}) \tag{14}$$

where we have neglected the term $(1+\kappa^{-1})$ in the second equation of eqs. (9). The solution of η_n are then constants fixed by the field boundary condition (10) to be

$$\eta_n = \left[\frac{\sinh(n+1)x}{\sinh x}\right]^2 - 1. \tag{15}$$

After perform the Fourier transformation on eqs. (5), we get the solution of the densities of λ n-strings,

$$\sigma_1 + \sigma_1^h = \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * [\rho + \sigma_2^h]$$
$$\sigma_n + \sigma_n^h = \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * [\sigma_{n+1}^h + \sigma_{n-1}^h]. \tag{16}$$

If we assume that σ_n and σ_n^h are independent of λ or let $c=0$, the total number of down isospins has the form,

$$\sum_n n\sigma_n = \frac{\rho}{2} - \frac{n_m+1}{2}\sigma_{n_m} e^{n_m\Omega/T} \tag{17}$$

where n_m is maximal length of λ strings. In the absence of Rabi field, we have $M/N = 1/2$, the system at high temperature is a quasi "paramagnetic" state. If the external field Ω is small, expanding eq. (17) for small filed x and integrating the equation over λ space, we get the magnetization of the model, let M_m be the total number of isospin rapidities in all n_m-strings,

$$\frac{S_z}{L} = \frac{M_m}{2L}\left(1 + \frac{n_m\Omega}{T} + \frac{n_m^2\Omega^2}{2T^2} + \cdots\right) \tag{18}$$

where the first term in the brackets arises from self-magnetization, while the others are contributed by Rabi field. Eq. (18) indicate that the magnetic property of the model in high temperature regime dominated by Curie's law $\chi \propto 1/T$.

Strong coupling limit: When $\eta \to \infty$, $K_n(k) = 0$, from eqs. (9) we have

$$\epsilon = k^2 - \Omega - \mu. \tag{19}$$

The free energy of the system (11) at low temperature now can be solved by step integration,

$$F/L = \mu D - \frac{2}{\pi}\left[\frac{1}{3}\mu^{3/2} + \frac{T^2\pi^2}{24\mu^{1/2}}\right] \tag{20}$$

where the external field is set to zero.

We can not deduce the specific heat directly from the free energy obtained above because the chemical potential is a function of temperature. From eqs. (5), the density of charge rapidity has the form

$$\rho = \frac{1}{2\pi}\frac{1}{1+e^{(k^2-\mu)/T}}. \tag{21}$$

Clearly, at zero temperature, the Fermi surface is just the square root of the chemical potential, so we have $\mu_0 = \pi^2 D^2$. At low temperature, however, it is determined by $D = N/L = \int \rho(k)dk$. After integration, we have a time dependent chemical potential

$$\mu = \mu_0 \Big[1 - \frac{\pi^2 T^2}{24\mu_0^2}\Big]^{-2}. \tag{22}$$

Then the free energy becomes

$$F/L = \mu_0 D\Big[1 + \frac{\pi^2 T^2}{12\mu_0^2}\Big] - \frac{2}{3\pi}\mu_0^{3/2}\Big[1 + \frac{\pi^2 T^2}{4\mu_0^2}\Big]. \tag{23}$$

Since by thermodynamics $\mathcal{S} = -\partial F/\partial T$ and $C_v = T\partial\mathcal{S}/\partial T$, we find the specific heat at low temperature is Fermi-liquid like

$$\mathcal{S} = C_v = \frac{T}{6D}. \tag{24}$$

It is the same as the result of one-component case, since for the strong coupling limit the isospin and the charge are decoupled completely, the contribution of isospin to the free energy vanishes.

Weak coupling limit: In order to discuss the possibility of the existence of BEC, we consider the problem in weak coupling limit $u \to 0$. And isospin-isospin reaches its maximal correlation. At low temperature, however, we do not take string hypothesis for simplicity. Because $\lim_{c\to 0} K_n(x) = \delta(x)$, together with eqs. (5) and eqs. (9), we obtain

$$\rho(k) = \frac{1}{2\pi}\frac{(3e^{\varepsilon_0} - 1)(e^{-\varepsilon_0} + 1)}{(3e^{2\varepsilon_0} + 1)(1 - e^{-\varepsilon_0})} \tag{25}$$

where $\varepsilon_0 = (k^2 - \mu)/T$. The positive definition of $\rho(k)$ require that the chemical potential is negative. As we known the density of scalar boson is $2\pi\rho = 1/(1 - e^{-\varepsilon_0})$ which prevents the BEC in 1D and 2D system because of the infrared divergence. However, the density function (25) still does not resolve this problem. Consequently, BEC does not happen in this model.

To summarize, we discussed the general thermodynamics of one dimensional SU(2) bosons with δ-function interaction by using the strategy of TBA. It was shown that the ground state is an isospin "ferromagnetic" state which differs from the ground state of 1D fermions, while at high temperature, it is "paramagnetic" state and the magnetic property is dominated by Curie's law. In strong coupling limit, we obtain the exact expression of the dependence of chemical potential, entropy and specific heat on temperature which are Fermi-liquid like, while in weak coupling limit, we found the infrared divergence of charge roots density function prevents the existence of BEC.

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